MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 11

Theorem 1. *(Magidor-Shelah) Suppose that $\langle \kappa_n \mid n < \omega \rangle$ in an increasing sequence of supercompact cardinals, $\kappa_\omega = \sup_n \kappa_n$ and $\mu = (\sup_n \kappa_n)^+$. Then the tree property holds at μ.*

Proof. Let T be a μ-tree. The proof has two rounds. First we will show that for some n, there is a "weak subtree" whose levels are of size at most κ_n. Then we will use the supercompactness of κ_{n+1} to construct an unbounded branch through that weak subtree. We will denote the nodes of the tree by $u = \langle \alpha, \delta \rangle$, where $\alpha < \mu$ indicates the height (i.e. level) of u and $\delta < \kappa_\omega$. In other words, for every $\alpha < \mu$, the α-th level of the tree is

$$T_\alpha = \{\langle \alpha, \delta \rangle \mid \delta < \kappa^{+\omega}\}.$$

Lemma 2. *There is an unbounded $I \subset \mu$ and $n < \omega$ such that for all $\alpha, \beta \in I$ with $\alpha < \beta$, there are $\xi, \delta < \kappa_n$ such that $\langle \alpha, \xi \rangle <_T \langle \beta, \delta \rangle$.*

Proof. Let $j : V \to M$ be a μ-supercompact embedding with critical point κ_0. Then $j(T)$ is a tree of height $j(\mu)$ and levels of size at most $j(\kappa_\omega)$. Note that $\rho := \sup j"\mu < j(\mu)$. Let $u \in j(T)_\rho$. Since $j(T)$ is a tree, for all $\alpha < \mu$, there is a node $u_\alpha \in j(T)_{j(\alpha)}$, such that $u_\alpha <_{j(T)} u$. Denote $u_\alpha = \langle j(\alpha), \xi_\alpha \rangle$, where $\xi_\alpha < j(\kappa_\omega) = \sup_n j(\kappa_n)$. Then there is some $n_\alpha < \omega$, such that $\xi_\alpha < j(\kappa_{n_\alpha})$. Let $f : \mu \to \omega$ be $f(\alpha) = n_\alpha$. Since μ is regular (both in V and M), there is some unbounded $I \subset \mu$ and $n < \omega$, such that $n = n_\alpha$ for all $\alpha \in I$.

We claim that I and n are as desired. For if $\alpha, \beta \in I$ with $\alpha < \beta$, then $u_\beta <_{j(T)} u$ and $u_\alpha <_{j(T)} u$. Since $j(T)$ is a tree, that implies that $u_\alpha <_{j(T)} u_\beta$. I.e. $M \models (\exists \xi, \delta < j(\kappa_n))(\langle j(\alpha), \xi \rangle <_{j(T)} \langle j(\beta), \delta \rangle)$. Then by elementarity, in V, there are $\xi, \delta < \kappa_n$ such that $\langle \alpha, \xi \rangle <_T \langle \beta, \delta \rangle$.

□

Now we are ready to show that there is a branch through the tree. Let Let $j : V \to N$ be a μ-supercompact embedding with critical point κ_{n+1}. Let $\gamma \in j(I)$ be such that $\gamma \geq \sup j"\mu$. Note that then for all $\alpha \in I$, $N \models j(\alpha), \gamma \in j(I); j(\alpha) < \gamma$. Then for all $\alpha \in I$ by elementarity,

$$N \models (\exists \xi, \delta < j(\kappa_n) = \kappa_n)(\langle j(\alpha), \xi \rangle <_{j(T)} \langle \gamma, \delta \rangle).$$

So, for all $\alpha \in I$, fix $\xi_\alpha, \delta_\alpha < \kappa_n$, such that $\langle j(\alpha), \xi_\alpha \rangle <_{j(T)} \langle \gamma, \delta_\alpha \rangle$. Consider $\alpha \mapsto \delta_\alpha$, which is a function from I to κ_n. Since μ is regular, there is some unbounded $J \subset I$ and $\delta < \kappa_n$, such that for all $\alpha \in J$, $\delta_\alpha = \delta$. Let $b := \{\langle \alpha, \xi_\alpha \rangle \mid \alpha \in J\}$.

Claim 3. *For all $\alpha < \beta$ with $\alpha, \beta \in J$, we have that $\langle \alpha, \xi_\alpha \rangle <_T \langle \alpha, \xi_\alpha \rangle$.*

Proof. Fix such α, β. Since, in N, $j(T)$ is a tree and

$$\langle j(\alpha), \xi_\alpha \rangle <_{j(T)} \langle \gamma, \delta \rangle, \text{ and } \langle j(\beta), \xi_\beta \rangle <_{j(T)} \langle \gamma, \delta \rangle,$$

we have $\langle j(\alpha), \xi_\alpha \rangle <_{j(T)} \langle j(\beta), \xi_\beta \rangle$. By elementarity, $\langle \alpha, \xi_\alpha \rangle <_T \langle \beta, \xi_\beta \rangle$. $\square$

Then $\{u \in T \mid (\exists v \in b) u <_T v\}$ is an unbounded branch through T.

$\square$

As discussed before, in order to have the tree property at κ^{++} for a singular strong limit κ, we must have failure of SCH at κ. So, to get the tree property at both κ^+ and κ^{++}, the next question of interest is obtaining the failure of SCH at κ together with the tree property at κ^+. Recall that weak square implies the existence of an Aronsjazn tree. Thus, we must first ask whether we can get failure of SCH at κ together with $\neg\square^*_\kappa$. This was solved in 2007 by Gitik and Sharon:

Theorem 4. *(Gitik-Sharon) Suppose κ is supercompact. Then there is a generic extension in which, $\mathrm{cf}(\kappa) = \omega$, SCH fails at κ and $\square^*_\kappa$ also fails.*

The proof of this theorem uses diagonal supercompact Prikry forcing. Let κ be supercompact, and let U be a normal measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$. For every n, let U_n be a normal measure on $\mathcal{P}_\kappa(\kappa^{+n})$ obtained by projecting U. I.e. for every $A \in U$, let $A \restriction n := \{x \cap \kappa^{+n} \mid x \in A\}$ and set $U_n := \{A \restriction n \mid A \in U\}$. Write $x \prec y$ to denote $x \subset y$ and $o.t.(x) = \kappa_y := \kappa \cap y$. We note that for every n, $\{x \mid \kappa_x \text{ is a cardinal }, o.t.(x) = \kappa_x^{+n}\} \in U_n$. So in the definition below assume that this is always the case.

Definition 5. $\mathbb{P}$ *consists of elements* $p := \langle x_0^p, ..., x_{n-1}^p, A_n^p, A_{n+1}^p, ...\rangle$, *where:*

1. *for* $i < n$, $x_i^p \in \mathcal{P}_\kappa(\kappa^{+i})$, *and* $x_i^p \prec x_{i+1}^p$,
2. *for* $i \geq n$, $A_i^p \in U_i$.

The length of p *is* $\mathrm{lh}(p) = n$, *and* $\mathrm{stem}(p) = \langle x_0^p, ..., x_{n-1}^p\rangle$. *For* p *as above,* $q \leq p$ *if:*

1. $\mathrm{lh}(q) \geq \mathrm{lh}(p)$, $\mathrm{stem}(p)$ *is an initial segment of* $\mathrm{stem}(q)$,
2. *for* $\mathrm{lh}(p) \leq i < \mathrm{lh}(q)$, $x_i^q \in A_i^p$,
3. *for* $i \geq \mathrm{lh}(q)$, $A_i^q \subset A_i^p$.

q *is a direct extension of* p, *denoted by* $q \leq^* p$, *if* $q \leq p$ *and* $\mathrm{lh}(q) = \mathrm{lh}(p)$.

Lemma 6. *(The Prikry property) For every condition* p *and sentence in the forcing language* ϕ, *there is a direct extension* $q \leq^* p$, *such that* q *decides* ϕ.

Corollary 7. $\mathbb{P}$ *does not add bounded subsets of* κ.

Proof. Suppose that $\tau < \kappa$ and $p \Vdash \dot{a} \subset \tau$. Then for every $\beta \in \tau$, by the Prikry property, let $q_\beta \leq^* p$ be such that q_β decides "$\beta \in \dot{a}$". Let q be such that $\mathrm{stem}(q) = \mathrm{stem}(p)$ and for all $n \geq \mathrm{lh}(q)$, $A_n^q = \bigcap_{\beta<\tau} A_n^\beta$. The latter is a measure one set since $\tau < \kappa$. So q is a condition which is stronger than every q_β. Then, in V define:

$$b = \{\beta < \tau \mid q \Vdash \beta \in \dot{a}\}.$$

It follows that $q \Vdash \dot{a} = b$.

That means that no bounded subsets can be added: if $a \in V[G]$ is a bounded subsets of κ, by the above argument there are densely many q's such that for some $b \in V$, $q \Vdash \dot{a} = b$. So there is such a q in G. So, $a = b \in V$. □

Corollary 8. *Cardinals up to and including κ are preserved.*

Lemma 9. *$\mathbb{P}$ had the $\kappa^{+\omega+1}$-chain condition.*

Proof. Any two conditions with the same stem are compatible, and there are at most $\kappa^{+\omega}$-many possible stems. It follows that any antichain has size at most $\kappa^{+\omega}$.

□

Corollary 10. *Cardinals greater than or equal to $\kappa^{+\omega+1}$ are preserved.*

Next we will show that the cardinals in between are all singularized to have cofinality ω, and $\kappa^{+\omega}$ is collapsed to κ. Let G be $\mathbb{P}$-generic, and let $s^* = \bigcup_{p\in G} \operatorname{stem}(p)$. Then $s^* = \langle x_n^* \mid n < \omega\rangle$, where every $x_n^* \in \mathcal{P}_\kappa(\kappa^{+n})$ and $x_n^* \prec x_{n+1}^*$.

Lemma 11. *Suppose that $\langle A_n \mid n < \omega\rangle \in V$ is such that every $A_n \in U_n$. Then for all large n, $x_n^* \in A_n$.*

Proof. Let $D = \{p \mid \forall n \geq \operatorname{lh}(p)(A_n^p \subset A_n)\}$. This is a dense set, because for any $p \in \mathbb{P}$, we can take $q \leq p$ in D, by setting $\operatorname{stem}(q) = \operatorname{stem}(p)$ and for all $n \geq \operatorname{lh}(p)$, $A_n^q = A_n^p \cap A_n$.

Let $p \in D \cap G$. Then for all $n \geq \operatorname{lh}(p)$, $x_n^* \in A_n$.

□

It turns out that this condition is not only necessary, but also sufficient for genericity. The proof is similar to the proof of the Prikry property.

Lemma 12. $\bigcup_n x_n = \kappa_V^{+\omega}$

Proof. Clearly $\bigcup_n x_n \subset \kappa_V^{+\omega}$. Suppose now that $\alpha < \kappa_V^{+\omega}$. Let $l < \omega$ be such that $\alpha < \kappa^{+l}$. For all $n \geq l$, let $A_n = \{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid \alpha \in x\} \in U_n$. By the above lemma for all large n, $x_n^* \in A_n$. That means that for all large n, $\alpha \in x_n^*$.

□

Let $\mu := \kappa_V^{+\omega+1}$.

Corollary 13. *In $V[G]$, for all $0 \leq n < \omega$, $\operatorname{cf}(\kappa_V^{+n}) = \omega$, and μ be comes the successor of κ*

Proof. Every x_k^* is a set in V of size less than κ (although of course the sequence $\langle x_n^* \mid n < \omega\rangle$ is not in V). So for every $n < \omega$,

$$\beta_k^n := \sup(x_k^* \cap \kappa_V^{+n}) < \kappa_V^{+n}.$$

Then in $V[G]$,

$$\kappa_V^{+n} = (\bigcup_k x_k) \cap \kappa_V^{+n} = \bigcup_k x_k \cap \kappa_V^{+n} = \sup_{k<\omega} \beta_k^n.$$

So, κ becomes a singular cardinal with cofinality ω, all of the κ_V^{+n}'s for $n > 0$ are collapsed to κ, and μ becomes the new cardinal successor of κ.

$\square$